Finite Spaces Tweaks and Twerks
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1 Introduction

By definition, finite topological spaces are topological spaces defined on sets that have only finitely many points in them. While the axioms for topological spaces were developed for principally for infinite spaces, when applied to finite spaces, they show surprisingly intriguing properties. This write-up accounts for some of these properties, while assuming that the reader has a basic idea about topological spaces.

2 Finite Spaces and Partial Orders

As stated earlier in the introduction, we inherit the three axioms of general topological spaces, i.e., a topological space on a set X is consisted of subsets of X such that, the whole set X and the empty set is open, the finite intersection of open sets is open, and arbitrary union of open sets is open. However, when these definitions are tweaked a bit, they start to show interesting properties when considering finite spaces.

2.1 Basic Properties of Finite Spaces

We first introduce a topological space by merits of Paul Alexandroff.

Definition 2.1. A topological space is A-space if the intersection of any family of open sets is open.

Lemma 2.2. A finite space is A-space.

Proof. If a topological space is finite, it can only have possibly finitely many intersections, which means arbitrary intersection of open sets are finite intersection of open sets, which are open under axiom of topological spaces. □

We then define some axioms for separation.

Definition 2.3. Consider a topological space (X, $\mathscr{T}$).
(i) X is T_0 if for any two points of X, there is an open neighborhood of one that does not contain

the other.
(ii) X is T_1 if each point of X is a closed subset.
(iii) X is T_2, or Hausdorff, if any two points of X have disjoint open neighborhoods.

Lemma 2.4. $T_2 \implies T_1 \implies T_0$.

Proof. Assume that X is a topological space and Hausdorff, or T_2. Then, for any two points of X, there exist disjoint open neighborhoods containing each point. Suppose $x \in X$, then, $\forall y \in X$, $y \neq x$, $\exists$ neighborhood V of each y s.t. $x \notin V$. Take the union of such neighborhoods. Then, $\{x\} = X \setminus \cup V$, which is closed. So $T_2 \to T_1$.
Assume now that X is T_1. Then every point of X is a closed subset. Consider $x, y \in X$, $x \neq y$. Then $\{x\}$ is a closed subset, and $y \in X \setminus \{x\}$, which is an open neighborhood not containing x. This shows that $T_1 \to T_0$. □

We all know discrete topology in general topological spaces for a set X: they are the collection of every subset of X. They turn out to behave quite peculiar in finite spaces.

Corollary 2.5. A finite space is T_1 if and only if it has the discrete topology.

Proof. Assume that X is a finite T_1 space. Then, every singleton set of X is closed. Consider $x \in X$. Since x is T1, $\{y\}$ is closed $\forall y \in X$, $y \neq x$. Take the union of such $\{y\}$. The axiom of topology says that arbitrary open sets are open, which means that arbitrary closed sets are closed. In this way, $\cup\{y\}$ is closed, so $\{x\} = X \setminus \cup\{y\}$ is open. This shows that X has the discrete topology.
Now assume X is a finite space with discrete topology. Then by definition, all singleton sets are open. Moreover, for arbitrary $x \in X$, $\{x\}$ is closed: the union of singleton sets $\{y\}$ $\forall y \in X$, $y \neq x$ is open, so $\{x\} = X \setminus \cup\{y\}$ is closed. This shows that X is T_1. □

If a finite space is T_1, then it carries the discrete topology; this is not generally true for general topological spaces: a common Hausdorff space (e.g. $\mathbb{R}^2$ need not carry the discrete topology, but every singleton set is closed due to its Hausdorff property). However, the converse is true for topological spaces in general.

2.2 Partial Order on Finite Spaces

We now introduce the partial order on finite spaces. This concept is important since it not only gives a unique minimal basis, but also defines equivalence classes of open sets.

Definition 2.6. Let X be a finite space. For $x \in X$, define U_x to be the intersection of the open sets that contain x. Define a relation $\leq$ on the set X by $x \leq y$ if $x \in U_y$, or, equivalently, $U_x \subset U_y$.

Lemma 2.7. The set of open sets U_x is a basis for X. Indeed, it is the unique minimal basis for X.

Proof. Let $U \subset X$, and $x \in U$ be arbitrary. Then $x \in U_x \subset U$ since U_x is the intersection of all open sets containing x. This proves that the set of U_x is a basis.
Let $\mathbb{B}$ be another basis. Then, $\forall\ x \in X$, $\exists$ a $B \in \mathbb{B}$ s.t. $x \in B \subset U_x$. But then $\forall\ B \in \mathbb{B}$ containing x, $U_x \subset B$. In this way, The set of U_x and $\mathbb{B}$ are equivalent, and this proves that the set of U_x is the unique minimal basis. □

Lemma 2.8. The relation $\leq$ is transitive and reflexive. It is a partial order if and only if X is T_0.

Proof. $\forall\ x \in X$, $U_x \subset U_x$.
If $x \leq y$, $y \leq z$, then $U_x \subset U_y$ and $U_y \subset U_z$. By set inclusion $U_x \subset U_z$, or $x \leq z$.
First assume that X is T_0 and $x \leq y$ and $y \leq x$. Assume the contrary that $x \neq y$. Since X is T_0, for any two points of X, there is an open neighborhood of one not containing the other. Without loss of generality, assume x has an open neighborhood not containing y. Since U_x is the minimal set, $y \notin U_x$, or $U_y \not\subset U_x$, a contradiction to $x \leq y$. Thus, $x = y$.
Conversely, assume X is of partial order. Then, for x, $y \in X$, $x \neq y$, either x, y are incomparable, or $x < y$ or $x > y$. If x, y are not comparable, then U_x, U_y would be the open sets containing each element but not the other, which means that X is Hausdorff, or T_2. By lemma 2.4, this implies that X is T_0. Without loss of generality, assume now that $x < y$. Then, $U_x \subsetneq U_y$, so $U_x \subset U_y$ but $U_y \not\subset U_x$. In this way, U_x is an open set containing x but not y, and X is T_0. □

Lemma 2.9. A finite set X with a reflexive and transitive relation $\leq$ determines a topology with basis the set of all sets $U_x = \{y | y \leq x\}$.

Proof. This follows directly from lemma 2.7. □

Proposition 2.10. For a finite set X, the topologies on X are in bijection with the reflexive and transitive relations $\leq$ on X. The topology corresponding to $\leq$ is T_0 if and only if the relation $\leq$ is a partial order.

Proof. The second statement follows directly from lemmas 2.7, 2.8, and 2.9.
To prove for the first statement, assume a finite space X and its topology $\mathscr{T}$ and topology induced by the reflexive and transitive relation $\mathscr{T}'$. Consider $f : \mathscr{T} \longrightarrow \mathscr{T}'$. Since the number of open sets remain unchanged under the reflexive and transitive relation, $|\mathscr{T}'| = |\mathscr{T}|$. By definition, the two topologies are in bijection with each other. □

3 Continuous Maps and Homeomorphisms

For me, at the least, the most amazing thing when it comes to continuous maps and homeomorphisms in terms of finite spaces is that some homeomorphism classes of finite spaces can be in bijective correspondence with some equivalence classes of matrices.
To introduce this result, we first present some lemmas.

Lemma 3.1. Consider finite spaces X, Y. A function $f : X \longrightarrow Y$, is continuous if and only if it is order preserving: i.e., $x \leq y$ in X implies $f(x) \leq f(y)$ in Y.

Proof. Assume $f : X \longrightarrow Y$ is continuous, and $x \leq y$. Then, $x \in U_y \subset f^{-1}(U_{f(y)})$, U_y being the minimal basis in X containing y and $U_{f(y)}$ being the minimal basis in Y containing $f(y)$. In this way, $f(x) \in U_{f(y)}$, or $f(x) \leq f(y)$. Conversely, assume $V \subset Y$ is open. $\forall\ f(y) \in V$, then $U_{f(y)} \subset V$. $\forall\ x \in U_y$, $x \leq y$, so $f(x) \leq U_{f(y)} \subset V$, and $x \in f^{-1}(V)$. In this way, $f^{-1}(V) = \cup U_y$ is open. □

Lemma 3.2. A map $f : X \longrightarrow X$ is a homeomorphism if and only if f is either one-to-one or onto.

Proof. If f is homeomorphism, then by definition it is one-to-one and onto. Assume the converse that f is either one-to-one or onto. Since f is a map from finite set X to itself, being either one-to-one or onto implies the other, so f is bijective. We now prove that f send open sets to open sets. Let U be an open subset of X, suppose the contrary that f(U) is not open. Then $X \setminus f(U)$ is open. By continuity, $f^{-1}(X \setminus f(U)) = f^{-1}(X) \setminus f^{-1}(f(U)) = X \setminus U$ is open, which is a contradiction. Since f is continuous, bijective, and sends open sets to open sets, f is a homeomorphism. □

Definition 3.3. Consider square matrices $M = \left(a_{i,j}\right)$ with integer entries that satisfy the following properties.
(i) $a_{i,i} \geq 1$.
(ii) $a_{i,j}$ is -1, 0, or 1 if $i \neq j$.
(iii) $a_{i,j} = -a_{j,i}$ if $i \neq j$.
(iv) $a_{i_1,i_s} = 0$ if there is a sequence of distinct indices $\{i_1, ..., i_s\}$ s.t. $s > 2$ and $a_{i_k,i_k+1} = 1$ for $1 \leq k \leq s-1$.
We say that such matrices M and N are equivalent if there is a permutation matrix T s.t. $T^{-1}MT = N$ and let $\mathscr{M}$ denote the set of equivalence classes of such matrices.

Theorem 3.4. *The homeomorphism classes of finite spaces are in bijective correspondence with $\mathscr{M}$. The number of sets in a minimal basis for X determines the size of the corresponding matrix, and the trace of the matrix is the number of elements of X.*

4 Connectivity and Path Connectivity

It is one of the classic results of point-set topology that path connectivity implies connectivity, but not necessarily vice versa. For finite spaces, however, this relation holds both ways.

4.1 General Topological Spaces

We first define connectivity and path-connectivity formally.

Definition 4.1. A space X is *connected* if it is not the disjoint union of two open, nonempty subsets. Equivalently, X is connected if the only clopen (open and closed) subsets of X are itself and the empty set. In addition, define an equivalence relation $\sim$ on X by $x \sim y$ if x and y are elements of some connected subspace of X. An equivalence class under $\sim$ is called a component of X.

Now we introduce some basic properties of connected spaces in general topological spaces.

Lemma 4.2. The components of X are connected, X is the disjoint union of its components, and any connected subspace of X is contained in a component.

Proof. Since the components of X are defined as equivalence classes, they are by disjoint and their union make up the whole of X.
To prove the third statement, let $A \in X$ be a connected subspace. Assume that A is in two components of X, C_1, C_2, and $x_1 \in A \cap C_1$, $x_2 \in A \cap C_2$. Since A is a connected space, $x_1 \sim x_2$, which implies that $C_1 = C_2$. □

Lemma 4.3. If $f : X \longrightarrow Y$ is a continuous map and X is connected, then $f(X)$ is a connected subspace of Y.

Proof. To prove the contrapositive of the statement, suppose that $f(X)$ is not a connected subspace of Y. Then, $\exists$ separating sets U, V s.t. U, V open, nonempty, $U \cup V = f(X)$, $U \cap V = \emptyset$. In this way, $f^{-1}(X) = f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(\emptyset) = \emptyset$. Since $f^{-1}(U) = \{x \in X | f(x) \in f(U)\}$ and U, V are nonempty, $f^{-1}(U)$, $f^{-1}(V)$ are nonempty. By continuity, since U, V are open, $f^{-1}(U)$, $f^{-1}(V)$ are open. In this way, $f^{-1}(U)$, $f^{-1}(V)$ separates X, hence proved. □

Definition 4.4. Let I = [0, 1] be equipped with the usual metric topology as a subspace of R^n. It is a connected space, so by lemma 4.2, so is its image. A map $p : I \longrightarrow X$ is called a *path* from p(0) to p(1) in X. A space X is *path connected* if any two points can be connected by a path. Define a second equivalence relation $\simeq$ on X by $x \simeq y$ if there is a path connecting x to y. An equivalence class under $\simeq$ is a path component of X.

To answer the question about the relation between connectedness and path connectedness, we first prove formally that a common example is in fact a connected space.

Lemma 4.5. The unit interval, I, is connected.

Proof. Assume that I is not connected. Then, $I = A \cup B$ with A, B open, disjoint, nonempty. Assume without loss of generality that $0 \in A$. Let $x = inf\{B\}$ (this is valid since every bounded subset of $\mathbb{R}$ has an infimum). So, $x \neq 0$ or $x \neq 1$. If $x = 0$, then $B = \{1\}$, which is closed, a contradiction. If $x = 0$, then since $0 \in A$, A is open in I, $[0, a) \subset A$ for some $a > 0$, contradicting $x = infB$. So, $x \in (0, 1)$. If $x \in A$, then $\exists$ an open neighborhood with $x \in (a, b) \subset A$. If $x \in B$, then $\exists$ an open neighborhood with $x \in (a, b) \subset B$. In both cases, it is a contradiction: $x = infB$ so I is closed. □

Theorem 4.6. *$x \sim y$ implies $x \simeq y$, but not conversely in general. i.e., for a topological space X, if X is path connected, then it is connected; but not generally vice versa.*

Proof. To prove that path connectivity implies connectivity, assume the contrary that X is path connected but not connected. Then $X = A \cup B$ with A, B open in X, disjoint, and nonempty. Take $x \in A$, $y \in B$. Let γ be path from x to y. Then, $I = \gamma^{-1}(A) \cup \gamma^{-1}(B)$, which is disjoint, nonempty, and open. This contradicts the fact that I is connected, and proves that I must be path-connected. Conversely, consider $\bar{S} = \{(x, sin(\frac{1}{x})|x \in (0, 1]\} = (S \cup \{\{0\} \times [-1, 1]\}) \subset \mathbb{R}^2$. Famously called the "topologist's sine curve", $\bar{S}$ is path connected but not connected. □

Lemma 4.7. The path components of X are path connected, X is the disjoint union of its path components, and any path connected subspace of X is contained in a path component. Each path component is contained in a component.

Proof. Similar to lemma 4.2. □

4.2 Finite Spaces

We now turn our attention to finite spaces in general.

Lemma 4.8. Each U_x is connected. If X is connected and $x, y \in X$, there is a sequence of points z_i, $1 \leq i \leq s$, s.t. $z_1 = x$, $z_s = y$ and either $z_i \leq z_{i+1}$ or $z_i \geq z_{i+1}$ for $i < s$.

Proof. Assume U_x is separated by A and B. Without loss of generality assume $x \in A$, but then by definition of U_x, $U_x \subset A$, which leaves $B = \emptyset$, a contradiction.
Fix $x \in X$, let $A \subset X$ be a set consisted of points y that are connected to x by some sequence z_i. Since $z \leq z'$ implies $U_z \subset U_{z'}$, A is open. For y is not connected to x, neither is any point of U_y, so the complement of A is open, or A is closed. $x \in A$, so A is nonempty. Since X is a connected space and A is a nonempty clopen subset, $A = X$. Since the entirety of X can be reconstructed by A, which is configured solely in terms of z_is, such sequencing of points does exist. □

Lemma 4.9. If $x \leq y$, then there is a path p connecting x and y.

Proof. Define a continuous map $p(t) = x$ if $t < 1$ and $p(1) = y$. Let V be an open subset of X. If $x \in V$ and $y \notin V$, then $p^{-1}(V) = [0, 1)$. If $y \in V$, then $p^{-1}(V) = I$. If $y \in V$, then $x \in V_y \subset U$ since $x \leq y$. Therefore, $f^{-1}(V) = I$. □

Proposition 4.10. A finite space is connected if and only if it is path connected.

Proof. Assume X is a finite, connected topological space. By lemma 4.8, $\forall\ x,\ y \in X$, $\exists$ some sequence z_i, $1 \leq i \leq s$ s.t. $z_1 = x$, $z_s = y$ s.t. either $z_i \leq z_{i+1}$ or $z_i \geq z_{i+1}$. By lemma 4.9, then, x, y are joined by a path.
The converse is true by theorem 4.6. □

There is a more general condition than finiteness of a set that induces an if-and-only-if relation between connectivity and path connectivity: local path connectivity. By definition, a space X is locally path connected at x if for every neighborhood U of x, there is a connected neighborhood V of x contained in V, and X is called a locally path connected space if this is true for every $x \in X$. Finite spaces are locally path connected by lemma 4.9, so finite spaces serves as an example for such an axiom.

5 Conclusion

Finite spaces, viewed in lens of separation axiom, continuity, and connectivity, share some characteristic with general topological spaces but provide some convenience for thinking of certain properties. Most notably, the topologies on a topological space X can be completely described by a well-defined transitive and reflective relation, the homeomorphism classes of finite topological spaces are in bijection with classes of matrices, and path connectivity is equivalent to connectivity for finite spaces.

6 References

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